Strings and matrix models on genus g Riemann Surfaces
old title:STRINGS AND LARGE N QCD [*]

Published in the proceedings of "Beyond the Standard Model", Ed. K. Milton, 1991

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ABSTRACT

Comment: This is a summary of old work on connections between discrete area preserving diffeomorphisms, reduced SU(N) Yang-Mills, strings, and the quantum Hall effect on a Riemann surface of genus g. It is submitted to the archives due to the interest expressed by colleagues who are currently working on matrix models, and who could not have access to the proceedings in which the article was published. The text that follows is the version published in 1991.
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In this talk I will describe an attempt to bridge string theory and large N QCD, as obtained in recent papers. This is based on a relation between area preserving diffeomorphisms and $SU(\infty)$. The reduced model of QCD takes the form of a version of string theory that is related to ordinary string theory in the gauge $det(\gamma) = -1$, where γ_{ij} is the world sheet metric.

1. Area Preserving Diffeomorphisms and $SU(\infty)$

Consider a compact Riemann surface parametrized by the Euclidean parameters $\sigma^i = (\tau, \sigma)$. The infinitesimal transformations $\sigma^i \rightarrow \sigma^i + \xi^i$ that leave the area element $d^2\sigma = d\tau d\sigma$ invariant are the area preserving diffeomorphisms. The parameters for such transformations satisfy $\partial_i \xi^i = 0$ and can be expressed in the form $\xi^i = \epsilon^{ij}\partial_j\Omega + \sum_r \lambda_r v_r^i$,

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where $\Omega(\tau, \sigma)$ is an arbitrary function (except for periodicity properties on the surface), λ_r are constants and $v_r^i(\tau, \sigma)$ are the harmonic 1-forms, of which there are as many as twice the genus of the surface, $r = 1, 2, \cdots 2g$. Corresponding to Ω we have the local subalgebra whose generator is labelled as $L(\tau, \sigma)$ and corresponding to λ_r we have the global translation operators K_r which generate translations along the $2g$ cycles on the surface. The algebra of these generators was given for the sphere [1] and torus [2]. The general commutation rules for any Riemann surface of genus g including the general form of a potential anomaly, was obtained in [3], where generalizations to supersymmetry and higher dimensions was also given. Here we will mainly concentrate on the local subalgebra of $L(\tau, \sigma)$.

The relation to $SU(\infty)$ was explicitly given on the sphere [1] and torus [2] and it has only recently been obtained for surfaces of any genus [4] as will be described here. First consider the torus. We expand in terms of a complete set of periodic functions on the torus, one writes $L(\vec{\sigma}) = \sum_{\vec{n}} L_{\vec{n}} \, exp(i\vec{n} \cdot \vec{\sigma})$, with $\vec{n} = (n, n')$, and obtain the Lie algebra of area preserving diffeomorphisms in Fourrier space as

$$[L_{\vec{n}}, L_{\vec{m}}] = i(\vec{n} \times \vec{m})L_{\vec{n}+\vec{m}} \tag{1}$$

where $(\vec{n} \times \vec{m}) = nm' - mn'$. This algebra is related to SU(N) as $N \to \infty$ as follows. Consider the $N \times N$ (N=odd) Weyl matrices h and g that satisfy $h^N = 1 = g^N$ and $gh = hg\omega$, where $\omega = exp(i4\pi/N)$. These matrices are explicitly given as $h = diag(1, \omega, \omega^2, \cdots, \omega^{N-1})$ and $g_i^j = \delta_{i+1}^j$ defined by identifying the indices $i \, or \, j = N+1 \to 1$, so that it has non-zero entries only above the diagonal and at the $(i, j) = (N, 1)$ location. There are N^2 linearly independent powers of these matrices, $h^n g^{n'}$, $n, n' = 0, 1, \cdots, (N-1)$, that are unitary and close under multiplication. Excluding the $n = 0 = n'$ identity matrix, the remaining ones are traceless and close under commutation. Thus, we construct the SU(N) generators in this basis $l_{\vec{n}} = \frac{N}{4\pi}h^n g^{n'} \omega^{nn'/2}$. They can be shown to satisfy the commutation rules

$$[l_{\vec{n}}, l_{\vec{m}}] = i\frac{N}{2\pi}sin(\frac{2\pi}{N}\vec{n} \times \vec{m}) \, l_{\vec{n}+\vec{m}}, \tag{2}$$

that parallels (1). In this form, taking the $N \to \infty$ limit (1) and (2) become identical, thus displaying the relation between area preserving diffeomorphisms and $SU(\infty)$ on the torus. Note, however, that this is true only if $(n \times m)/N$ is small, which means the relationship between $SU(\infty)$ and area preserving diffeomorphisms can be valid only when integrated with an appropriate set of functions. This is analogous to the equivalence between classical mechanics and quantum mechanics in the limit of $\hbar \to 0 \, \sim (\frac{1}{N} \to 0)$, provided we use an appropriate set of wavefunctions. A similar construction on the sphere uses the spherical harmonic basis Y_{jm} [1].

For a genus g surface, following ref.[4] we consider $SU(N)$ with $N = N_1 \times N_2 \times \cdots N_g$. We label the N-dimensional fundamental representation by a composite index $\psi_{i_1 i_2 \cdots i_g}$ where $i_1 = 1, 2, \cdots N_1$; $i_2 = 1, 2, \cdots N_2$; $i_g = 1, 2, \cdots N_g$. We see that we can construct

the subgroups $SU(N_1)$, $SU(N_2)$, $\cdots$, $SU(N_g)$ in a direct product basis

$$l_{\vec{n}_1}^{(N_1)} \times 1_{N_2} \times \cdots \times 1_{N_g} \qquad 1_{N_1} \times l_{\vec{n}_2}^{(N_2)} \cdots \times 1_{N_g} \qquad 1_{N_1} \times 1_{N_2} \times \cdots \times l_{\vec{n}_g}^{(N_g)} \qquad (3)$$

where $l_{\vec{n}_k}^{(N_k)}$ is a $N_k \times N_k$ matrix constructed from g, h matrices of rank N_k and satisfies (2). We can then construct the full $SU(N)$ by taking all possible $N \times N$ matrices of the direct product form (C_N is a constant, see below)

$$l_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g} = C_N \left(h^{n_1} g^{n_1'} \times h^{n_2} g^{n_2'} \times \cdots \times h^{n_g} g^{n_g'} \right) exp\big(i2\pi \sum_i \frac{n_i n_i'}{N_i}\big) \qquad (4)$$

We can show that these $N \times N$ matrices satisfy the following matrix product rules

$$l_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g} \, l_{\vec{m}_1 \vec{m}_2 \cdots \vec{m}_g} = C_N \, exp\big(i2\pi \sum_{i=1}^{g} \frac{\vec{n}_i \times \vec{m}_i}{N_i}\big) \, l_{\vec{n}_1 + \vec{m}_1, \vec{n}_2 + \vec{m}_2, \cdots, \vec{n}_g + \vec{m}_g} \qquad (5)$$

and the commutation rules

$$[l_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}, l_{\vec{m}_1 \vec{m}_2 \cdots \vec{m}_g}] = 2iC_N \, sin\big(i2\pi \sum_{i=1}^{g} \frac{\vec{n}_i \times \vec{m}_i}{N_i}\big) \, l_{\vec{n}_1 + \vec{m}_1, \vec{n}_2 + \vec{m}_2, \cdots, \vec{n}_g + \vec{m}_g} \qquad (6)$$

which generalize the $SU(N)$ commutation rules of (2) to arbitrary genus.

It can be shown [4] that the $SU(N)$ transformations generated by $l_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}$ are closely related to discrete area preserving transformations of a lattice Riemann surface of genus g. This connection is obtained through generalized Jacobi theta functions defined on the genus g surface. A particular set of these functions is identified with our labeling of the N-dimensional fundamental representation above

$$\psi_{i_1 i_2 \cdots i_g} = Theta \begin{bmatrix} \frac{i_1}{N_1} & \frac{i_2}{N_2} & \cdots & \frac{i_g}{N_g} \\ 0 & 0 & \cdots & 0 \end{bmatrix} (\vec{Z}(z); \Omega) \times F(z, \bar{z}) \qquad (7)$$

where $z = \tau + i\sigma$ is a point on the Riemann surface, Ω_{ij} is the $g \times g$ period matrix, $Z_i(z) = \int_{z_0}^{z} \omega_i(z')dz'$ is the Jacobi variety and $\omega_i(z)$ are the Abelian differentials. As is well known when integrated around the standard α_i, β_i cycles one has $\int_{\alpha_j} \omega_i = \delta_{ij}$, $\int_{\beta_j} \omega_i = \Omega_{ij}$. We now make a lattice by dividing the α_j, β_j cycles into N_j (not necessarily equal) intervals $\Delta\alpha_j, \Delta\beta_j$ such that the integration for each interval gives $\int_{\Delta\alpha_j} \omega_i = \frac{1}{N_i}\delta_{ij}$, $\int_{\Delta\beta_j} \omega_i =$

$\frac{1}{N_j}\Omega_{ij}$. This implies that when z is translated by n_j intervals along α_j and n'_j intervals along β_j we get a transformation on the Riemann surface of the form

$$z' = z + \sum_j n_j \Delta\alpha_j + \sum_j n'_j \Delta\beta_j, \qquad Z_i(z') = Z_i(z) + \frac{n_i}{N_i} + \Omega_{ij}\frac{n'_j}{N_j} \qquad (8)$$

Using the properties of the theta function we can now show explicitly that our N-dimensional basis undergoes the transformation

$$\psi_{i_1 i_2 \cdots i_g}(z') = \frac{1}{C}\left(l_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}\right)^{j_1 j_2 \cdots j_g}_{i_1 i_2 \cdots i_g}\psi_{j_1 j_2 \cdots j_g}(z) \qquad (9)$$

The factor $F(z, \bar{z})$ in (7) is inserted to cancel the well known extra phase that appears in the transformation of the theta function when (8) is applied. We have thus demonstrated that translations on the N^2 points of the lattice Riemann surface are expressed by our $SU(N)$ generators given in eqs.(4-6,9). Taking linear combinations of all these translations with arbitrary continuous coefficients gives the full $SU(N)$.

It is worth mentioning that, up to a z-dependent factor, our wavefunctions $\psi_{i_1 i_2 \cdots i_g}$ form the basis of linearly independent solutions to the problem of a charged particle moving on a Riemann surface of genus g in the presence of a magnetic field [5]. The $SU(N)$ symmetry is then identified with the magnetic translation group. This provides an approach for studying the generalization of the quantum Hall effect problem for arbitrary Riemann surfaces [5]. This problem also connects to the solutions of topological quantum field theories in 2+1 dimensions.

We may now ask how to relate our original area preserving transformations $L(\tau, \sigma)$ to the matrices $l_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}$ as $N \to \infty$? We postulate position-momentum like structures $Q_i(\tau, \sigma), P_i(\tau, \sigma)$ constructed on the Riemann surface of genus g, with Poisson brackets $\{Q_i, P_j\} = \partial_\tau Q_i \partial_\sigma P_j - \partial_\sigma Q_i \partial_\tau P_j = \delta_{ij}$. For example for the torus $Q = \tau, P = \sigma$. Next we construct the basis functions $f_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}(\tau, \sigma) = exp(i\sum_i n_i Q_i + n'_i P_i)$ which satisfy the Poisson brackets

$$\{f_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}, f_{\vec{m}_1 \vec{m}_2 \cdots \vec{m}_g}\} = \sum_{i=1}^{g}(\vec{n}_i \times \vec{m}_i)\ f_{\vec{n}_1 + \vec{m}_1, \vec{n}_2 + \vec{m}_2, \cdots, \vec{n}_g \vec{m}_g}. \qquad (10)$$

Then we can write $L(\tau, \sigma) = \sum L_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g} f_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}$ where the operators $L_{\vec{n}_1 \vec{n}_2 \cdots \vec{n}_g}$ satisfy commutation rules with the same structure constants as eq.(10), which is the $N \to \infty$ limit of eq(6) provided we choose $N_i = N^{1/g}$ and $C = \frac{N^{1/g}}{4\pi}$. This yields the generalization of eqs.(1,2) and of the discrete Riemann surface analysis given previously for the torus [6].

2. Strings from large N QCD

Large N theories including QCD may be analyzed by using reduced Eguchi-Kawai models. Reduced models are known to reproduce the planar graphs. We have suggested that in a double scaling limit reduced QCD could describe the sum over all genus [6]. In the reduced model the original NxN matrix gauge field $(A_\mu)_i^j(x^\mu)$ is replaced by the same field at the single space-time point $x^\mu = 0$. Derivatives are replaced by a commutator with a fixed matrix $(P_\mu)_i^j$ that plays the role of the translation generator. The covariant derivative becomes $a_\mu = P_\mu + A_\mu \sim iD_\mu$. This leads to the Yang-Mills field strength

$$(F_{\mu\nu}) = [iD_\mu, iD_\nu] \rightarrow [a_\mu, a_\nu]_i^j \equiv (f_{\mu\nu})_i^j, \tag{11}$$

and the reduced gauge theory action and path integral [8,9,10]

$$S_{red} = -\frac{1}{4}\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{g_d^2(\Lambda)} Tr(f_{\mu\nu}f^{\mu\nu}) \qquad \int \prod_\mu Da_\mu f(a_\mu)\, exp(iS_{red}(a)) \tag{12}$$

where Λ is a cutoff and $f(a) = \int \prod_\mu DU_\mu \delta(a_\mu - U_\mu P_\mu U_\mu^\dagger)$, where U_μ is a unitary matrix corresponding to a reduced Wilson line integral [10].

We wish to rewrite the large-N reduced model as a theory defined on Riemann surfaces reminiscent of string theory [6,7]. We illustrate this for the torus, however using the formalism of the previous section the approach is immediately generalized to any genus. We expand the matrix $(a_\mu)_i^j = C_1 \sum (l_{\vec{n}})_i^j a_\mu^{\vec{n}}$. The constant C_1 is a normalization factor. Similarly, let us consider the gauge field for area preserving diffeomorphisms reduced to a single space-time point. Since the adjoint representation is labelled by the continuous variables τ, σ this gauge field is labelled as $A_\mu(\tau, \sigma)$. This looks like a string field $X_\mu(\tau, \sigma)$ defined on the Riemann surface. In order to make it suggestive we will label our gauge potential as X_μ. For the torus it may be expanded as $X_\mu(\tau, \sigma) = C_2 \sum a_\mu^{\vec{n}} exp(i\vec{\sigma} \cdot \vec{n})$ where the coefficients have been labelled as $a_\mu^{\vec{n}}$ in order to establish a parallel with the coeffients of expansion for the matrix $(a_\mu)_i^j$ above. C_2 is a normalization constant which defines the normalization of X_μ relative to $a_\mu^{\vec{n}}$. Next compare the field strengths for the reduced SU(N) theory and the reduced area preserving gauge theory. The adjoint action of the area preserving diffeomorphism group (for any surface) is defined by the commutation rules (1), so that the adjoint representation is expressed by a Poisson bracket in the τ, σ variables. This yields the field strength $F_{\mu\nu} = \{X_\mu, X_\nu\}(\vec{\sigma}) \equiv \epsilon^{ij}\partial_i X_\mu \partial_j X_\nu$, which is nothing but the *string area element* for any surface. Thus, for the torus we have

$$(f_{\mu\nu})_i^j = [a_\mu, a_\nu]_i^j = C_1^2 \sum a_\mu^{\vec{n}} a_\nu^{\vec{m}} \frac{N}{2\pi} sin(\frac{2\pi}{N}\vec{n} \times \vec{m})\, (l_{\vec{n}+\vec{m}})_i^j,$$
$$F_{\mu\nu}(\vec{\sigma}) = \{X_\mu, X_\nu\}(\vec{\sigma}) = C_2^2 \sum a_\mu^{\vec{n}} a_\nu^{\vec{m}} (\vec{n} \times \vec{m})\, exp(i(\vec{n} + \vec{m}) \cdot \vec{\sigma}). \tag{13}$$

We now see that as $N \rightarrow \infty$ (for a suitable behavious of $a_\mu^{\vec{n}}$ that allows the replacement $sin(\frac{2\pi}{N}\vec{n} \times \vec{m}) \rightarrow \vec{n} \times \vec{m}$ in the sums), the trace $Tr(f_{\mu\nu}f^{\mu\nu})$ will produce the same

expression as the integral $\int d^2\sigma F_{\mu\nu} F^{\mu\nu}$ except for an overall constant. Thus, we find that as $N \to \infty$ the reduced action (12) takes the form

$$S_{red} = -\frac{(2\pi/\Lambda)^{d-4}}{g_d^2(\Lambda)} \left(\frac{NC_1}{2C_2\Lambda}\right)^4 \int d^2\sigma F_{\mu\nu} F^{\mu\nu}(\vec{\sigma}). \tag{14}$$

This Lagrangian is interpreted as the square of the area element spaned by the string X_μ. It looks different than standard string action, however if we write the string action in the gauge $det(\gamma) = -1$ with $\gamma_{ij} = \partial_i X \cdot \partial_j X$, then $S \sim \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ij} \partial_i X \cdot \partial_j X = \int d\tau d\sigma\, det(\partial_i X \cdot \partial_j X)$, so (14) is seen to be identical to the string action semi-classically. The quantum path integral for this form of string theory needs to be defined by introducing a cutoff. We may regard the finite N version of eq.(12) as the cutoff version of this string theory. Indeed, we have argued elsewhere [6] that this corresponds to taking a lattice Riemann surface with N^2 points. The presence of the non-trivial factor $f(a)$ in the measure (12) indicates that the string version of QCD differs from standard string theory at the quantum level.

The action (12) yields the planar graphs in the standard limit (i.e. $N \to \infty$ first and then $\Lambda \to \infty$). However, we may also consider a corelated way of taking the limit in a way analogous to the recent double scaling limit that yields the sum over all genus in recent investigations of 2-dimensional gravity. Since the coupling constant $g_d(\Lambda)$ is really a function of Λ, this would suggest that in order to achieve the analog of the double scaling limit we would have to send $\Lambda \to \infty$ in an N-dependent fashion. This discussion leads to the following picture. The path integral of the reduced model in matrix form is now expected to yield a path integral over the string variable DX_μ as well as a *sum over surfaces* of genus g, just as in string theory. $\sum_g \int Dm DX_\mu f(X)\, exp(S_{red}^{(g)}(X))$, where $f(X)$ is the measure in (12) rewritten in terms of the string variable X_μ.

While this form is strongly reminiscent of string theory in the gauge $det(\gamma) = -1$, the measure in the path integral does not look quite the same. We really are discussing the dynamics of flux tubes of gauge theories rather than standard string theory. It may be useful to further study the consequences of (12) or (15) by using string techniques in order to learn non-perturbative properties of gauge theories and in particular of QCD in the confinement region. A place to start is d=2. This will be investigated in the future.

Can our observations be useful in the usual string theory? In particular, can we use it to sum over all genus and discuss non-perturbative string physics? To some extent this has to do with the measure being different. However, nothing stops us from going back to the reduced matrix action of (2) and simply change the measure in the path integral, so that it would be compatible with the required measure in string theory in the gauge $det(\gamma) = -1$. So, this provides us now with a matrix model which may describe the sum over all genus in string theory!!

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